SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 15

MATH 241

12.5.1

Proof. If $\lambda > 0$, we get $y(t) = C_1 \cos\sqrt{\lambda}t + C_2 \sin\sqrt{\lambda}t$. $y'(t) = -\sqrt{\lambda}C_1 \sin\sqrt{\lambda}t + \sqrt{\lambda}C_2 \cos\sqrt{\lambda}t$. Let $t = 0$, we get $\sqrt{\lambda}C_2 = y'(0) = 0$ so $C_2 = 0$.

$$y(1) + y'(1) = C_1 \cos\sqrt{\lambda} - C_1\sqrt{\lambda}\sin\sqrt{\lambda}$$

This is zero if and only if $\cos\sqrt{\lambda} = \sqrt{\lambda}\sin\sqrt{\lambda}$. so $\sqrt{\lambda}$ is the solution to $\alpha = \cot\alpha$, the eigenvalues are the square of the solutions to $\alpha = \cot\alpha$, and for each eigenvalue λ, the eigenfunction is (up to a multiple of constant) $\cos\sqrt{\lambda}t$. □

12.5.5

Proof. For each eigenvalue λ, the square norm of $\cos\sqrt{\lambda}t$ on $[0, 1]$ is

$$\int_0^1 \cos(\sqrt{\lambda}t)^2 dt = \int_0^1 \frac{\cos 2\sqrt{\lambda}t + 1}{2} dt = \frac{1}{2} + \frac{\sin 2\sqrt{\lambda}}{4\sqrt{\lambda}}$$

If you want, you can further simplify by plugging in $\sqrt{\lambda} = \cot\sqrt{\lambda}$ and $\sin 2\sqrt{\lambda} = 2\sin\sqrt{\lambda}\cos\sqrt{\lambda}$ to get $\|y_\lambda(t)\|^2 = \frac{1}{2} + \frac{2\sin\sqrt{\lambda}\cos\sqrt{\lambda}}{4\cot\sqrt{\lambda}} = \frac{1}{2} + \frac{1}{2}\sin^2\sqrt{\lambda}$. □

12.5.7

Proof.

$$x^2y'' + xy' + \lambda y = 0$$

This is a special kind of ODE called Cauchy-Euler equation. This was taught in Math 240, but we give a hint on how to solve it in case someone did not take that course. If you consider the change of variable $x = e^t$, the chain rule tells you $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = e^t\frac{dy}{dx} = x\frac{dy}{dx}$. Moreover,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(\frac{dy}{dt}) = \frac{d}{dt}(x\frac{dy}{dx}) = \frac{dx}{dt}\frac{d}{dx}(x\frac{dy}{dx}) = e^t[\frac{dy}{dx} + x\frac{d^2y}{dx^2}] = x\frac{dy}{dx} + x^2\frac{dy^2}{dx^2}$$

Therefore the first two terms $x^2y'' + xy'$ is just $\frac{d^2y}{dt^2}$. Therefore the equation becomes $\frac{d^2y}{dt^2} + \lambda y = 0$. The boundary conditions should also be changed, since when $x = 1$ this means $t = 0$, when $x = 5$ this means $t = \ln 5$. Therefore the Sturm-Liouville problem becomes $\frac{d^2y}{dt^2} + \lambda y = 0, y(0) = 0, y(\ln 5) = 0$. This reduces to the regular Sturm-Liouville problem. The eigenvalues are $\lambda = \frac{n^2\pi^2}{(\ln 5)^2}$, the eigenfunctions are $y_n = \sin\frac{n\pi}{ln5}t = \sin(\frac{n\pi}{ln5}\ln x)$.

If we first change it into a standard form as a second order linear ODE, it is

$$y'' + \frac{1}{x}y' + \lambda\frac{1}{x^2}y = 0$$

therefore the integral factor is $e^{\int\frac{1}{x}dx} = e^{\ln x} = x$, so multiply x to this equation we get $xy'' + y' + \lambda\frac{1}{x}y = 0$. This is equal to $(xy')' + \lambda\frac{1}{x}y = 0$, in the standard self-adjoint form of a Sturm-Liouville problem, and $p(x)$ is $\frac{1}{x}$, so the orthogonality relation is, for any $n \neq m$, $\int_1^5 \frac{1}{x}\sin(\frac{n\pi}{ln5}\ln x)\sin(\frac{m\pi}{ln5}\ln x)dx = 0$. □

Remark Note that although we solve the equation by substitution in t, we still need to look up for the orthogonality in x.

12.5.11

Proof. $p(x) = \frac{1}{1+x^2}$, so the orthogonality is given as for any $n \neq m$, $\int_0^1 \frac{1}{1+x^2}\sin(4n\arctan x)\sin(4m\arctan x)dx = 0$.

That has nothing to do with the substitution of variables. But when you want to find the eigenvalues and eigenfunctions, some trick that is similar to 12.5.7 is needed. If you let $x = \tan\theta$, then $\frac{dx}{d\theta} = \frac{1}{\cos^2\theta} = \frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta} = \tan^2\theta + 1 = x^2 + 1$. Then it follows

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = (x^2+1)\frac{dy}{dx}$$

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(\frac{dy}{d\theta}) = \frac{d}{d\theta}((x^2+1)\frac{dy}{dx}) = \frac{dx}{d\theta}\frac{d}{dx}((x^2+1)\frac{dy}{dx}) = (1+x^2)\frac{d}{dx}((x^2+1)\frac{dy}{dx})$$

Now look at our equation, by multiplying $(1+x^2)$ to both sides we get

$$(1+x^2)\frac{d}{dx}((x^2+1)\frac{dy}{dx}) + \lambda y = 0$$

therefore the equation is finally changed into $\frac{d}{d\theta}(\frac{dy}{d\theta}) + \lambda y = 0$, or equivalently

$$\frac{d^2y}{d\theta^2} + \lambda y = 0$$

This is again the standard form we know about.

The boundary conditions are translated as $y(0) = 0$ and $y(\frac{\pi}{4}) = 0$ since when $x = 0, 1$, $\theta = 0, \frac{\pi}{4}$. Therefore the eigenvalues are $(\frac{n\pi}{\frac{\pi}{4}})^2 = 16n^2$, eigenfunctions are $y(t) = \sin 4n\theta = \sin(4n\arctan x)$.

□

Fall 10, #12

Proof. If $\lambda > 0$, $y(t) = C_1 \cos \sqrt{\lambda} t + C_2 \sin \sqrt{\lambda} t$, $y'(t) = -C_1 \sqrt{\lambda} \sin \sqrt{\lambda} t + C_2 \sqrt{\lambda} \cos \sqrt{\lambda} t$. Therefore $y'(0) = 0$ implies $C_2 = 0$, $y'(\pi) = 0$ implies $C_1 \sqrt{\lambda} \sin \sqrt{\lambda} \pi = 0$. Therefore $\sqrt{\lambda} = n$, $\lambda = n^2$ are the eigenvalues. Eigenfunctions are $C_1 \cos nt$. **Note**: plug in the formula for $y(t)$, not $y'(t)$!

If $\lambda = 0$, $y(t) = C_1 t + C_2$, so $y'(t) = C_1$, therefore $C_1 = 0$. So C_2 solves the Sturm-Liouville problem. Therefore $\lambda = 0$ is an eigenvalue, with eigenfunction constant function C_2.

If $\lambda < 0$, $y(t) = C_1 \cosh \sqrt{-\lambda} t + C_2 \sinh \sqrt{-\lambda} t$, $y'(t) = C_1 \sqrt{\lambda} \sinh \sqrt{\lambda} t + C_2 \sqrt{\lambda} \cosh \sqrt{\lambda} t$. Therefore $y'(0) = 0$ implies $C_2 = 0$, $y'(\pi) = 0$ implies $C_1 \sqrt{\lambda} \sinh \sqrt{\lambda} \pi = 0$, but this is impossible since $\sqrt{\lambda} \pi \neq 0$. So we do not have negative eigenvalues.

To summarize, the eigenvalues are $n^2, n \in \mathbb{N}$(including $n = 0$), and the eigenfunctions are $y_n(t) = C \cos nt$. This unifies the two formula when the eigenvalue is positive or zero. □

Fall 09, #6

Proof. The same trick as dealing with Cauchy-Euler equation, let $x = e^t$, then the equation becomes $\frac{d^2 y}{dt^2} + 25\lambda y = 0$, the boundary conditions are $y'(0) = 0$, $y(1) = 0$. The general solution will have the form $C_1 \cos(\sqrt{25\lambda} t) + C_2 \sin(\sqrt{25\lambda} t)$, the conditions imply $C_2 = 0$, $\cos \sqrt{25\lambda} = 0$, so $\lambda = \frac{(2n+1)^2 \pi^2}{100}$ are the eigenvalues. It is left to yourselves to show you do not have a zero or negative eigenvalue.

The eigenfunctions are $C \cos(n + \frac{1}{2})\pi t = C \cos((n + \frac{1}{2})\pi \ln x)$. □